CHAPTER 7
Frequency Response (P.244, Ch.8)
Outline of Chapter 7

✓ Introduction
✓ Bode Plots (Logarithmic plots)
✓ Direct Polar Plots
✓ Nyquist’s Stability Criterion
✓ Phase Margin and Gain Margin and Their Relation to Stability
✓ ……..
Harry Nyquist (1889-1976)
- He received a Ph.D. in physics at Yale University in 1917.
- He developed the Nyquist stability criterion for feedback systems in the 1930s.
- He worked at AT&T's Department of Development and Research from 1917 to 1934, and continued when it became Bell Telephone Laboratories in that year, until his retirement in 1954.
Hendrik Wade Bode

(1905 –1982)

- an American engineer, researcher, inventor, author and scientist, of Dutch ancestry. As a pioneer of modern control theory and electronic telecommunications
- He received his B.A Degree in 1924, M.A, 1926, Ohio State University
- Sponsored by Bell Lab., he successfully completed his Ph.D, in physics in 1935.
- In 1938, he developed his asymptotic phase and magnitude plots.

Frequency Response: 1. Introduction

The two basic methods for predicting and adjusting a system’s performance:

- The root-locus methods

  **Advantage:**
  - It displays the location of the closed-loop poles as a parameter (gain) is varied
  - It explicitly gives information about transient performance

  The actual time response is easily obtained by means of the inverse Laplace transform because the precise root locations are known.

  **Disadvantages:**
  - It does not provide steady state performance information
  - It cannot handle constraints on frequency ranges for measurement noise and disturbance rejection

  *Need a new tool to address these issues*
The frequency-response method — a good complement to Root Locus

✓ We can infer stability and performance, namely steady state performance, from the same plot
✓ It is easy to include constraints on frequency ranges
✓ Permit evaluation of the effect of noise, exclude the noise by designing a passband and therefore improve the system performance
✓ Can use measured data when no model is available (also good for system identification)
✓ Time delays are handled correctly, contrary to root locus
✓ Analysis and design techniques are graphical (asymptotic lines) and easy to use
In this chapter:

- Two graphical representation of transfer functions are presented:
  - the logarithmic plots
  - the polar plots.

- Developing the Nyquist’s stability criterion using these plots.
- The performance of frequency domain.
1. Introduction: Frequency characteristic

Considering a system

\[
\frac{Y(s)}{X(s)} = G(s)
\]

System block diagram

If the input is

\[
x(t) = X \sin \omega t
\]

\[
X(s) = \frac{X\omega}{s^2 + \omega^2} = \frac{X\omega}{(s + j\omega)(s - j\omega)}
\]
1. Introduction: Frequency characteristic

\[ X(s) = \frac{X\omega}{s^2 + \omega^2} = \frac{X\omega}{(s + j\omega)(s - j\omega)} \]

\[ G(s) = \frac{A(s)}{B(s)} = \frac{A(s)}{(s - s_1)(s - s_2) \cdots (s - s_n)} \]

\[ Y(s) = G(s)X(s) \]

\[ = \frac{A(s)}{(s - s_1)(s - s_2) \cdots (s - s_n)} \cdot \frac{X\omega}{(s + j\omega)(s - j\omega)} \]

\[ = \frac{b}{s + j\omega} + \frac{\bar{b}}{s - j\omega} + \frac{a_1}{s - s_1} + \frac{a_2}{s - s_2} + \cdots + \frac{a_n}{s - s_n} \]
1. Introduction: Frequency characteristic

\[
Y(s) = G(s)X(s) = \frac{b}{s+j\omega} + \frac{\bar{b}}{s-j\omega} + \frac{a_1}{s-s_1} + \frac{a_2}{s-s_2} + \ldots + \frac{a_n}{s-s_n}
\]

\[
y(t) = be^{-j\omega t} + \bar{b}e^{j\omega t} + a_1 e^{s_1 t} + a_2 e^{s_2 t} + \ldots + a_n e^{s_n t}
\]

For a stable system, all transient components will approach to zero when time \(t\) goes infinity, only steady-state response keep down.

\[
y_\omega(t) = \lim_{t \to \infty} y(t) = be^{-j\omega t} + \bar{b}e^{j\omega t}
\]

Where \(b\) can be found by residues theorem or other method.
Where $G(j \omega)$ is a complex number, can be expressed by

\[
b = G(s) \frac{X\omega}{(s + j\omega)(s - j\omega)} \cdot (s + j\omega) \bigg|_{s = -j\omega} = -\frac{G(-j\omega)X}{2j}
\]

\[
\bar{b} = G(s) \frac{X\omega}{(s + j\omega)(s - j\omega)} \cdot (s - j\omega) \bigg|_{s = j\omega} = \frac{G(j\omega)X}{2j}
\]

Similarly, $G(-j\omega)$:

\[
G(-j\omega) = |G(-j\omega)| \cdot e^{-j\phi(\omega)} = |G(j\omega)| e^{-j\phi(\omega)}
\]

\[
y_\omega(t) = \lim_{t \to \infty} y(t) = be^{-j\omega t} + \bar{b}e^{j\omega t}
\]
1. Introduction: Frequency characteristic

\[ y_\omega (t) = b e^{-j\omega t} + b^* e^{j\omega t} = -|G(j\omega)| e^{-j\phi(\omega)} \cdot \frac{X e^{-j\omega t}}{2j} + |G(j\omega)| \cdot e^{j\phi(\omega)} \cdot \frac{X e^{j\omega t}}{2j} \]

\[ = |G(j\omega)| X \cdot \frac{e^{j(\omega t + \phi)} - e^{-j(\omega t + \phi)}}{2j} \]

\[ = |G(j\omega)| X \cdot \sin(\omega t + \phi) \]

or

\[ y_\omega (t) = Y \sin(\omega t + \phi) \]

Which is the magnitude of steady state response.

Where

\[ G(j\omega) = |G(j\omega)| e^{j\angle G(j\omega)} = |G(j\omega)| e^{j\phi(\omega)} \]

is usually called frequency characteristic (---frequency transfer function 3.3).

There are two methods to obtain G(j\omega): analyzing and experiment.
1. Introduction: Frequency characteristic

**Definition:**

The frequency response of a system is described as the steady-state response with a sine-wave forcing function for all values of frequency.

**Such as:**

**Input:** \( u_i(t) = A \sin \omega t \)

**Output:** \( u_o(t) = A \cdot M(\omega) \sin[\omega t + \alpha(\omega)] \)

\( M(\omega) \) The magnitude ratio of the frequency response

\( \alpha(\omega) \) The angle of the frequency response

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When a linear system with transfer function \( G(s) \) is excited by a sinusoid of frequency \( \omega \), the steady state output is a sinusoid of the same frequency, with magnitude scaled by \(|G(j \omega)|\) and phase shifted by \( \angle G(j \omega) \).
Introduction

1. Introduction: Frequency characteristic

The transfer function of linear and stable system:

\[ G(s) = \frac{B(s)}{A(s)} = \frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \]

The frequency response:

\[ M(\omega) = \frac{|B(j\omega)|}{|A(j\omega)|} = \frac{|K(j\omega-z_1)\cdots(j\omega-z_m)|}{|j\omega-p_1|\cdots|j\omega-p_m|} = |G(j\omega)| \]

\[ \alpha(\omega) = \angle B(j\omega) - \angle A(j\omega) = \angle G(j\omega) \]

\[ = \angle K + \angle(j\omega-z_1) + \cdots + \angle(j\omega-z_m) - \angle(j\omega-p_1) - \cdots - \angle(j\omega-p_n) \]

See Section 4.12
Once the frequency response of a system has been determined, the time response can be determined by inverting the corresponding Fourier transform.

\[ R(j\omega) = \int_{-\infty}^{\infty} r(t)e^{-j\omega t} dt \]

\[ C(j\omega) = \Phi(j\omega)R(j\omega) = \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)}R(j\omega) \]

\[ c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(j\omega)e^{j\omega t} d\omega \]
1. Introduction: Frequency characteristic

The frequency domain plots belong to two categories:

1) The plot of the magnitude of the output-input ratio vs. frequency in rectangular coordinates and the corresponding phase angle vs. frequency. For example

\[ M(\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{a}{\sqrt{\omega^2 + a^2}} \]

\[ \alpha(\omega) = \angle \frac{Y(j\omega)}{X(j\omega)} = -\arctan \frac{\omega}{a} \]

In logarithmic coordinates these are known as bode plots or log magnitude diagram and phase diagram.

See Section 4.12
For a given sinusoidal input signal, the input and steady-state output are of the following forms:

\[ r(t) = R \sin \omega t \quad c(t) = C \sin(\omega t + \alpha) \]

the closed-loop frequency response is given by

\[ C(j\omega) = \frac{G(j\omega)}{R(j\omega)} = \frac{M(\omega) \angle \alpha(\omega)}{1 + G(j\omega)H(j\omega)} \]

The Frequency-response characteristics of \( C(j\omega)/R(j\omega) \) in rectangular coordinates

See P.247, Fig.8.1
2) The output-input ratio is plotted in polar coordinates with frequency as a parameter. Generally used only for the open-loop response and commonly referred to as Nyquist plots.

Example 7-1: The transfer function is $G(s)$

$$G(s) = \frac{1}{1+Ts}$$

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1 - j\omega T}{1+(T\omega)^2}$$

$$\left[ \text{Re}G(j\omega) - \frac{1}{2} \right]^2 + \text{Im}^2 G(j\omega) = \left( \frac{1}{2} \right)^2$$

The Nyquist plots is a arc of which the center of the circle is $(0.5, j0)$ and the radius is 0.5
The log magnitude and the angle are combined into a single curve of log magnitude vs. angle, with frequency as a parameter. This curve is called the Nichols chart or the log magnitude-angle diagram.

Example 7-2: the transfer function is $G(s)$

$$G(s) = \frac{1}{s(Ts + 1)}$$

$$G(j\omega) = \frac{1}{j\omega(j\omega T + 1)} = \frac{-T}{1 + \omega^2 T^2} - \frac{j}{\omega(1 + \omega^2 T^2)}$$

The log magnitude and the angle are combined into a single curve of log magnitude vs. angle, with frequency as a parameter. This curve is called the Nichols chart or the log magnitude-angle diagram.
1. Introduction: Frequency characteristic

Frequency Response

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